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Particle creation and vacuum polarisation in isotropic universe

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Abstract. The particle interpretation of quantised fields in homogeneous isotropic space–time is given which is based on the diagonalisation of the Hamiltonian constructed via the metrical stress–energy tensor (SET). This interpretation and the regularisation procedure of Zeldovich and Starobinsky allows to obtain the total renormalised vacuum expectation values of SET which include both vacuum polarisation and non-local terms describing the creation of particles. The case of the explicitly soluble cosmological model with initial singularity which is asymptotically static in the future is considered. The results of the calculation of the total SET in realistic Friedman cosmological models are presented.

1. Introduction

Recent developments in astrophysics gave a high priority to the problem of quantum effects in gravity. The most important of them is the creation of particles by strong gravitational fields near the cosmological singularity or in the process of gravitational collapse. This effect is connected with the problem of vacuum polarisation in an external gravitational field.

Particle creation in cosmological models has been under extensive investigation during the past decade (see, e.g., Parker 1969, Grib and Mamayev 1969, Zeldovich and Starobinsky 1971, Grib *et al* 1976, Mamayev *et al* 1976a, Frolov *et al* 1976, Mamayev *et al* 1976b, 1977, Chitre and Hartle 1977, Hu and Parker 1978, Audretsch and Schäfer 1978). On the other hand, a number of papers appeared recently (e.g. Dowker and Critchley 1976, Davies *et al* 1977, Bunch and Davies 1977, Bunch 1978a,b, Christensen 1978, Wald 1978, Davies 1978) where the initially divergent vacuum expectation values of the stress-energy tensor (SET) of quantised fields in curved space–times were calculated with the help of various regularisation schemes without appealing to any particle concept.

Up to now there has been a troublesome lack of contact between the two above mentioned directions. The main reason for it is that in studying particle creation mainly nonlocal terms in matrix elements of SET were being computed. In contrast, vacuum expectations of SET in the papers mentioned above were calculated either for massless fields or in the adiabatic limit, when the particle creation may be ignored.

Here we shall present a method of calculation of the total vacuum expectations of SET in isotropic space–times which takes into account both local vacuum polarisation and nonlocal terms describing the creation of particles. In § 2 the notation is introduced and the interpretation of the quantised field in terms of particles based on the diagonalisation of the Hamiltonian is constructed. The regularisation procedure for vacuum expectation values of the total SET is given in § 3. In § 4 an explicitly soluble cosmological model with initial singularity which is asymptotically static at *future* infinity is considered. Finally, in § 4 the results of the calculation of the total SET in realistic Friedman cosmological models are presented.

Units $\hbar = c = 1$ are used throughout the paper.

2. Particle interpretation of the quantised field

We shall consider charged spin-0 and -$\frac{1}{2}$ fields in homogeneous isotropic space–times with the metric:

$$ds^2 = a^2(\eta)[d\eta^2 - \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta] \tag{1}$$

where $\gamma_{\alpha\beta}$ is the metric on a 3-space with constant curvature $\varkappa = +1, 0$ or -1 (the spin-0 field is supposed to be conformally coupled to gravitation).

The field operator may be expanded in creation and annihilation operators

$$\phi = \int d\mu(J)[\phi_J^{(-)}(x)a_J^{(-)} + \phi_J^{(+)}(x)a_J^{(+)}] \tag{2}$$

where $\phi_J^{(\pm)}$ is a complete orthonormal set of solutions of the wave equations, defined as having positive (negative) frequency at the moment $\eta = \eta_0$ (or $\eta \to -\infty$).

The time dependence of $\phi_J(x)$ is described by a function $g_\lambda(\eta)$ which obeys an oscillatory equation; e.g. in the spin-0 case

$$\ddot{g}_\lambda(\eta) + \omega^2(\eta)g_\lambda(\eta) = 0, \qquad \omega^2(\eta) = \lambda^2 + m^2a^2(\eta) \tag{3}$$

(λ is the momentum quantum number). Evidently, the dependence of ω^2 in (3) on η leads to the frequency mixing, which in quantum theory means the particle creation. In non-stationary space–time no invariant concept of positive- and negative-frequency modes is available and particle interpretation cannot be constructed without imposing some additional principle.

As such a principle we adopt the requirement that the creation and annihilation operators must diagonalise the so called metrical Hamiltonian

$$H^{(s)} = \int_\Sigma d\sigma^i \zeta^k T_{ik}^{(s)} \tag{4}$$

where $T_{ik}^{(s)}$ is the conformal SET operator for the spin-s field, Σ is a spacelike hypersurface and ζ^k is the conformal Killing vector field (Grib and Mamayev 1969, Grib *et al* 1976). Hamiltonian (4) is a generator of conformal transformations of the metric (1) (Chernikov and Tagirov 1968, Brout *et al* 1978).

For $\eta > \eta_0$ (4) is a non-diagonal bilinear form in operators $a_J^{(\pm)}$ defined in (2). Perform a time-dependent Bogoljubov transformation

$$\begin{aligned} a_J^{(-)} &= \alpha_\lambda^*(\eta)b_J^{(-)}(\eta) - \beta_\lambda(\eta)b_J^{(+)}(\eta), \\ a_J^{(+)} &= \alpha_\lambda(\eta)b_J^{(+)}(\eta) \mp \beta_\lambda^*(\eta)b_J^{(-)}(\eta) \end{aligned} \tag{5}$$

with

$$|\alpha_\lambda|^2 \mp |\beta_\lambda|^2 = 1, \qquad \alpha_\lambda(\eta_0) = 1, \qquad \beta_\lambda(\eta_0) = 0$$

(upper sign corresponds to spin-0, lower sign to spin-$\frac{1}{2}$ field). Diagonalisation of (4) is achieved if the real bilinear combinations of α_λ and β_λ,

$$s_\lambda = |\beta_\lambda|^2, \qquad u_\lambda - \mathrm{i}v_\lambda = \pm 2\alpha_\lambda\beta_\lambda \exp\left(-2\mathrm{i}\int_{\eta_0}^{\eta} \omega \, \mathrm{d}\eta\right) \tag{6}$$

satisfy the following first-order differential equations (Zeldovich and Starobinsky 1971, Mamayev *et al* 1977)

$$\begin{aligned} \dot{s}_\lambda &= \tfrac{1}{2}w^{(s)}u_\lambda, \qquad \dot{v}_\lambda = 2\omega u_\lambda, \\ \dot{u}_\lambda &= w^{(s)}(1 \pm 2s_\lambda) - 2\omega v_\lambda \end{aligned} \tag{7}$$

with

$$w^{(0)} = m^2 a\dot{a}/\omega^2, \qquad w^{(1/2)} = m\dot{a}\lambda/\omega^2 \tag{8}$$

in spin-0 and -$\frac{1}{2}$ cases respectively. Initial conditions for equations (7) are $s_\lambda(\eta_0) = u_\lambda(\eta_0) = v_\lambda(\eta_0) = 0$.

The vacuum state at the time η is defined by equations

$$b_J^{(-)}(\eta)|0_\eta\rangle = b_J^{*(-)}(\eta)|0_\eta\rangle = 0. \tag{9}$$

The Heisenberg operators defined as

$$c_J^{(\pm)}(\eta) = b_J^{(\pm)}(\eta) \exp\left(\pm\mathrm{i}\int_{\eta_0}^{\eta} \omega(\eta')\, \mathrm{d}\eta'\right) \tag{10}$$

satisfy the Heisenberg equations of motion

$$\mathrm{d}c_J^{(\pm)}/\mathrm{d}\eta = \tfrac{1}{2}w^{(s)}c_J^{(\mp)} + \mathrm{i}[H^{(s)}, c_J^{(\pm)}]. \tag{11}$$

As is evident from (11), there are two sources of time dependence of operators $c_J^{(\pm)}$. The commutator accounts for the usual time evolution of Heisenberg operators, while the first term on the right-hand side of equation (11) describes the explicit time dependence due to the redefinition of the notion of particle for every moment η.

It is easy to show that the coefficients α_λ and β_λ of the transformation (5), which diagonalises the Hamiltonian, are connected with the solutions $g_\lambda(\eta)$ of equation (3) by the relations

$$\begin{aligned} g_\lambda &= \omega^{-1/2}[\alpha_\lambda^* \,\mathrm{e}^{\mathrm{i}\Theta} + \beta_\lambda \,\mathrm{e}^{-\mathrm{i}\Theta}], \\ \dot{g}_\lambda &= \mathrm{i}\omega^{1/2}[\alpha_\lambda^* \,\mathrm{e}^{\mathrm{i}\Theta} - \beta_\lambda \,\mathrm{e}^{-\mathrm{i}\Theta}] \end{aligned} \tag{12}$$

where $\Theta(\eta) = \int_{\eta_0}^{\eta} \omega(\eta')\, \mathrm{d}\eta'$. Similar relations hold for the spin-$\frac{1}{2}$ field.

Here is a point of contact between the present approach and that used by Zeldovich and Starobinsky (1971) and Hu and Parker (1978). In those papers the construction of observables was based directly on the decomposition of the solutions of the field equations into positive- and negative-frequency parts according to (12) without reference to the Hamiltonian diagonalisation. Obviously, the latter gives physical foundation for such a decomposition in the isotropic case.

Using equations (5) and (7) we may calculate expectation values of various physical quantities in the initial vacuum $|0\rangle = |0_{\eta_0}\rangle$. In particular, s_λ of (6) gives the spectrum of quasiparticles defined by operators $c_J^{(\pm)}$:

$$s_\lambda = \langle 0|c_J^{*(+)}c_J^{(-)}|0\rangle.$$

3. Regularisation of the stress–energy tensor

We now procede with computing the expectations of the SET operator in the initial vacuum state $|0\rangle$. Straightforward calculation gives (see e.g DeWitt 1975)

$$\langle 0|T_{ik}^{(s)}|0\rangle = \int d\mu(J) T_{ik}^{(s)}\{\phi_J^{(+)}, \phi_J^{(-)}\} \tag{13}$$

where $T_{ik}^{(s)}\{\cdot,\cdot\}$ is the bilinear form defined by the classical expression for the SET.

It is well known that (13) diverges even in Minkowsky space–time, where it is made finite (actually zero) by normal ordering. In the present case the leading (quartic) divergence of (13) may also be eliminated by normal ordering of T_{ik} in terms of time-dependent operators $c_J^{(\pm)}(\eta)$ (Mamayev *et al* 1976a). This normal ordering amounts to subtraction of zero-point oscillations of the vacuum $|0_\eta\rangle$:

$$N_\eta(T_{ik}) = T_{ik} - \langle 0_\eta|T_{ik}|0_\eta\rangle. \tag{14}$$

Note that the expectation value $\langle 0|N_\eta(T_{ik})|0\rangle$ is covariantly conserved since both $\langle 0|T_{ik}|0\rangle$ and $\langle 0_\eta|T_{ik}|0_\eta\rangle$ enjoy this property (modewise).

After a somewhat lengthy but straightforward calculation the result may be presented in the form of integrals over the momentum quantum number λ:

$$\begin{aligned}\langle 0|N_\eta(T_{00}^{(s)})|0\rangle &= [(2s+1)/\pi^2 a^2]\int d\mu^{(s)}(\lambda)\omega s_\lambda,\\ \langle 0|N_\eta(T_{\alpha\alpha}^{(s)})|0\rangle &= [(2s+1)\gamma_{\alpha\alpha}/3\pi^2a^2]\int (d\mu^{(s)}(\lambda)/\omega)(\lambda^2 s_\lambda - \tfrac{1}{2}f^{(s)}u_\lambda)\end{aligned} \tag{15}$$

($f^{(0)} = m^2a^2$, $f^{(1/2)} = \lambda ma$; the off-diagonal components are zero; $d\mu^{(0)}(\lambda) = \lambda^2\, d\lambda$, $d\mu^{(1/2)}(\lambda) = (\lambda^2 - \varkappa/4)\, d\lambda$). In the closed model ($\varkappa = +1$) one should replace $\int d\lambda \to \Sigma_\lambda$ where λ runs over $1, 2, \ldots$ in the spin-0 case and $\frac{3}{2}, \frac{5}{2}, \ldots$ in the spin-$\frac{1}{2}$ case.

In the spin-0 case one finds from equation (7) $s_\lambda \sim \lambda^{-6}$, $u_\lambda \sim \lambda^{-4}$ as $\lambda \to \infty$, so the integrals in (15) converge. In the spin-$\frac{1}{2}$ case, however, $s_\lambda \sim \lambda^{-4}$, $u_\lambda \sim \lambda^{-3}$ and expressions (15) diverge logarithmically. Moreover, in the general anisotropic case $s_\lambda \sim \lambda^{-2}$, u_λ, $v_\lambda \sim \lambda^{-1}$ and $\langle 0|N_\eta(T_{ik}^{(s)})|0\rangle$ diverge quadratically. Thus an additional regularisation is needed.

As an analysis with the Fock–Schwinger–DeWitt proper-time method shows (DeWitt 1975), the divergent part of the effective Lagrangian in the external gravitational field is purely local and contains terms, proportional to $\sqrt{-g}$, $\sqrt{-g}R$, $\sqrt{-g}(R_{ik}R^{ik} - \frac{1}{3}R^2)$ and $\sqrt{-g}R^2$. The normal ordering (14) kills the first divergency and is equivalent to the renormalisation of the cosmological term. In conformally flat space–time and for conformally coupled fields the divergencies of the third and of the fourth types in the SET do not arise. Neither is there a $\sqrt{-g}R$ term for the conformally coupled scalar field, whereas for the spin-$\frac{1}{2}$ field this term accounts for the above mentioned logarithmic divergency in (15), which is proportional to G_{ik} (Mamayev and Mostepanenko 1978a,b). In the general case the divergencies of all the four types are present and must be eliminated by means of some regularisation procedure.

The most suitable one in the case when the mode functions are known is the procedure proposed by Zeldovich and Starobinsky (1971). It amounts to substituting in (15) the quantities $s_\lambda - s_2 - s_4$, $u_\lambda - u_2 - u_4$ instead of s_λ, u_λ where s_n, u_n are the nonzero terms of order ω^{-n} in asymptotic expansions of the solutions of equations (7) in inverse powers of ω. Thus, in the $\varkappa=0,-1$ case, we obtain for the spin-0 field

$$\langle T_{00}^{(0)}\rangle_{\text{reg}}=(1/\pi^2a^2)\int \mathrm{d}\lambda\lambda^2\omega[s_\lambda-s_2-s_4] \tag{16}$$

and for the spin-$\frac{1}{2}$ field

$$\langle T_{00}^{(1/2)}\rangle_{\text{reg}}=(2/\pi^2a^2)\int \mathrm{d}\lambda\omega[(\lambda^2-\tfrac{1}{4}\varkappa)(s_\lambda-s_2)-\lambda^2 s_4] \tag{17}$$

(similar expressions may easily be obtained for $\langle T_{\alpha\alpha}^{(s)}\rangle_{\text{reg}}$). The term $\varkappa s_4/4$ is missing in (17) since it is of the order of $(ma)^{-2}$ and no terms including an inverse power of mass should be subtracted in a regularisation procedure (DeWitt 1975, Christensen 1978).

In the closed model ($\varkappa=+1$) the spectrum of λ is discrete and additional 'topological' terms must be included in (16), (17) (see, e.g., Ford 1976, Mamayev *et al* 1976a):

$$\Delta\langle T_{00}^{(s)}\rangle_{\text{reg}}=(-1)^{2s}\frac{2s+1}{2\pi^2a^2}\left[\sum_{\lambda=1/2}^{\infty}-\int_0^\infty \mathrm{d}\lambda\right]\frac{\mathrm{d}\mu^{(s)}(\lambda)}{\mathrm{d}\lambda}\omega. \tag{18}$$

The terms (18) appear because a discrete sum is subtracted in (14) whereas it should be an integral over λ since all the divergencies are local.

As was shown by DeWitt (1975), this procedure (as well as the 'adiabatic regularisation' of Parker and Fulling (1974)) gives the same results as the proper time method. Its advantage, however, is that, together with the particle interpretation described above, it gives a constructive method for obtaining the total SET including the nonlocal terms corresponding to particle creation.

For homogeneous isotropic space–times one finds

$$s_2=\tfrac{1}{16}(w^{(s)}/\omega)^2,$$
$$s_4=-\frac{1}{32\omega}\frac{w^{(s)}}{\omega}\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{w^{(s)}}{\omega}\right)\right]+\frac{1}{64}\left[\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{w^{(s)}}{\omega}\right)\right]^2\pm\frac{3}{256}\left(\frac{w^{(s)}}{\omega}\right)^4 \tag{19}$$

(u_2, u_4 may be obtained through (7)).

As stated above, in the spin-0 case, equations (15) give already finite expressions, while in the spin-$\frac{1}{2}$ case it is sufficient to replace s_λ and u_λ by $s_\lambda-s_2$, $u_\lambda-u_2$. However if one views the isotropic model as a limiting case of a more general anisotropic one the expectation of SET should be computed by means of equations (16) and (17).

The validity of the regularisation procedure (16), (17) is supported also by the following observation. The metric (1) with $\varkappa=-1$ and $a(\eta)\propto e^\eta$ describes part of the flat Minkowsky space enclosed in the future light cone of the origin. Hence it is a natural requirement that $\langle 0|T_{ik}|0\rangle_{\text{reg}}$ in this metric be zero if the vacuum $|0\rangle$ is chosen to coincide with the usual Minkowsky space vacuum. The calculation according to (16) and (17) shows (Starobinsky 1978) $\langle 0|T_{ik}|0\rangle_{\text{reg}}$ to be actually zero in this case, although the number of quasiparticles is nonzero (see Chitre and Hartle 1977). In fact the contribution of quasiparticles to $\langle T_{ik}\rangle_{\text{reg}}$ is cancelled by the vacuum polarisation part.

The contribution of the s_4 term in (16) and (17) is independent of the field mass m and hence is nonzero in the $m\to 0$ limit. At the same time if $m=0$, $s_\lambda=s_2=0$ and no particle creation occurs. Thus the s_4 term may be attributed to vacuum polarisation. Equations (16)–(18) give for $m\to 0$, $\varkappa=0,+1$

$$\langle T_{ik}^{(s)}\rangle_{\text{reg},0}=\frac{1}{1440\pi^2}[A_s^{(3)}H_{ik}+B_s^{(1)}H_{ik}] \tag{20}$$

where ${}^{(1)}H_{ik}$, ${}^{(3)}H_{ik}$ are the covariantly conserved tensors, quadratic in R_{ik} (see, e.g. Davies *et al* 1977), $A_0 = 1$, $B_0 = -1/6$, $A_{1/2} = 11/2$, $B_{1/2} = -1/2$. In the case $\varkappa = -1$ the third term $c_s J_{ik}/240\pi^2 a^2$ with $J_{ik} = (1 \oplus \frac{1}{3}\gamma_{ik})$, $c_0 = -1$, $c_{1/2} = -17/4$ must be added to (20). This is in agreement with the results of Davies *et al* 1977, Davies 1978. SET (20) has a nonzero trace (the so called conformal anomaly). Thus we see that in the conformally flat metric (1) there is a nonzero vacuum polarisation of massless fields. This may be considered as an effect of spontaneously broken conformal symmetry.

4. An explicitly soluble model

In this section we shall examine an explicitly soluble case of a model metric with flat 3-space and the scale factor

$$a(t) = a_+[1 - \exp(-2\gamma t/a_+)] \tag{21}$$

or, in terms of the conformal time η,

$$a(\eta) = (a_+/2)(\tanh \gamma\eta + 1). \tag{22}$$

Its initial expansion (for $t \ll t_* = a_+/\gamma$) is similar to that of the Milne universe: $a(t) \simeq 2\gamma t$, while for $t \gg t_*$ it becomes asymptotically static: $a(t) \to a_+$.

For brevity we shall consider here only the spin-0 case. The oscillatory equation (3), with initial conditions ensuring the vacuum state at $\eta \to -\infty$ $(t = 0)$, may be solved in terms of the hypergeometric function:

$$g_\lambda(\eta) = \lambda^{-1/2} e^{i\lambda\eta}(1 + e^{2\gamma\eta})^\tau F(a, b; c; -e^{2\gamma\eta}) \tag{23}$$

with

$$a = \tau + (i/2\gamma)(\omega_+ + \lambda), \qquad b = \tau + (i/2\gamma)(\omega_+ - \lambda),$$
$$\tau = \tfrac{1}{2}[1 - (1 - (mt_*)^2)^{1/2}], \qquad c = 1 + (i/\gamma)\lambda, \qquad \omega_+^2 = \lambda^2 + m^2 a_+^2.$$

From (12) we find

$$\alpha_\lambda = \frac{i}{2\omega^{1/2}} e^{i\Theta}(\dot{g}_\lambda^* - i\omega g_\lambda^*), \qquad \beta_\lambda = (i/2\omega^{1/2}) e^{i\Theta}(\dot{g}_\lambda - i\omega g_\lambda) \tag{24}$$

which allows us to obtain explicit expressions for s_λ, u_λ, v_λ through (24).

Since as $t \to \infty$ the expansion eventually ceases, the quasiparticles defined in § 1 become real particles; their spectrum is given by $n(\lambda) = \lim_{\eta\to\infty} s_\lambda(\eta)$ and may be obtained by analytic continuation of $F(a, b; c; -e^{2\gamma\eta})$ in the region $\eta > 0$:

$$n(\lambda) = \frac{\cosh[\pi(\omega_+ - \lambda)/\gamma] + \cos[\pi(1 - (mt_*)^2)^{1/2}]}{2 \sinh(\pi\omega_+/\gamma) \sinh(\pi\lambda/\gamma)}. \tag{25}$$

In the case $mt_* \gg 1$ (25) reduces to

$$n(\lambda) = \frac{1}{\exp(2\pi\lambda/\gamma) - 1} + \frac{1}{2\sinh(\pi\lambda/\gamma)} \exp[-(\pi/\gamma)(\omega_+ - ma_+)]. \tag{26}$$

Note that the first term of (26) for $\lambda \gg ma_+$ has the form of a black-body radiation spectrum with the temperature $kT = \gamma/(2\pi a_+)$.

The other limiting case is $mt_* \ll 1$ (the expansion takes a much shorter time than the Compton time m^{-1}); we have from (25)

$$n(\lambda) = \frac{\sinh^2[\pi(\omega_+ - \lambda)/2\gamma]}{\sinh(\pi\omega_+/\gamma)\sinh(\pi\lambda/\gamma)}. \tag{27}$$

For large but finite values of t the analytically continued hypergeometric function in (23) may be expanded into power series in $\exp(-2\gamma\eta)$ and then equations (24) and (6) give

$$s_\lambda(t) \simeq n(\lambda) + [(ma_+)^4\gamma^2/4\omega_+^6]\exp(-4t/t_*) \tag{28}$$

where we have omitted oscillatory terms which give neglectable contribution into integrals over the whole spectrum.

Next we shall calculate the stress-tensor of the field for $t \gg t_*$. Taking advantage of the fact that for the spin-0 field the integrals of s_λ, s_2 and s_4 in (16) converge, we may compute them separately. The contribution from s_4 coincides with (20); it is exponentially small for $t \gg t_*$ (as $\exp(-8t/t_*)$) and may be neglected. The second term of (28) and the term s_2 (see (19)) cancel each other. Thus for the energy density $\epsilon = \langle T_0^0 \rangle_{\text{reg}}$ we have

$$\epsilon \simeq (1/\pi^2 a_+^4)\int_0^\infty \mathrm{d}\lambda\, \lambda^2 \omega_+ n(\lambda). \tag{29}$$

In a similar way we obtain for the pressure $P = -\langle T_\alpha^\alpha \rangle_{\text{reg}}$ the result

$$P \simeq \frac{1}{3\pi^2 a_+^4}\int_0^\infty \frac{\mathrm{d}\lambda \lambda^4}{\omega_+} n(\lambda). \tag{30}$$

These quantities may be estimated in two cases: $t_* \gg m^{-1}$ and $t_* \ll m^{-1}$. For $t_* \gg m^{-1}$ using (26) we have

$$\epsilon \simeq \frac{2\zeta(3)}{\pi^5} m^4 (mt_*)^{-3}, \qquad P \simeq \frac{24\zeta(5)}{\pi^7} m^4 (mt_*)^{-5}$$

where $\zeta(z)$ is the zeta function. Obviously here we have a nonrelativistic gas of created particles ($P \ll \epsilon$); local vacuum polarisation terms have been neglected.

For $mt_* \ll 1$ the quantity $n(\lambda)$ is given by (27); integration according to (29), (30) yields

$$\epsilon \simeq \frac{m^4}{16\pi^2}\left[\ln\left(\frac{1}{\pi m t_*}\right) + \frac{3}{4}\right],$$

$$P \simeq \frac{m^4}{48\pi^2}\left[\ln\left(\frac{1}{16\pi m t_*}\right) + \frac{11}{4}\right].$$

In this case the created particles form an ultrarelativistic gas obeying equation of state $P \simeq \epsilon/3$.

5. Calculation of SET in the Friedman cosmological models

Let us now consider a more realistic case of a cosmological model of the Friedman type. Since the creation of particles is most intense near the singularity at times $t \sim m^{-1}$, we may take the expansion law to be

$$a(\eta)=a_1\eta^p=a_0t^q, \qquad 0<q<1. \tag{31}$$

In order to exclude the influence of the period $0<t<t_{pl}\sim 10^{-43}$ s (of which we have no information) assume that at $t<t_{pl}$ the scale factor $a(\eta)$ goes to zero in a sufficiently smooth manner. Solving equations (7) and evaluating the integrals in (16) for spin-0 field in the quasi-euclidean or hyperbolic space–time ($\varkappa=0,-1$) we have:
for $t\ll m^{-1}$

$$\epsilon^{(0)}=\frac{m^4}{16\pi^2}\left(\ln\frac{1}{mt}+D^{(0)}\right)-\frac{q^2m^2}{48\pi^2t^2}+\frac{F^{(0)}}{480\pi^2t^4} \tag{32}$$

for $t\gg m^{-1}$

$$\epsilon^{(0)}=K^{(0)}m^4(mt)^{-3q}+\frac{F^{(0)}}{480\pi^2t^4} \tag{33}$$

where $F^{(0)}=q^2(2q^2-6q+3)$ and $D^{(0)}$, $K^{(0)}$ are some other constants depending on q (see Mamayev *et al* 1976a). The expressions for the pressure may be obtained from (32), (33) through the conservation equation $\dot\epsilon=-3\dot a(\epsilon+P)/a$.

The first term in (33) gives the contribution of the created particles while the second one corresponds to the vacuum polarisation.

For a realistic spherical model ($\varkappa=+1$) $ma(t)\gg 1$ for $t>t_{pl}$. Therefore the corresponding corrections (18) to equations (32), (33) in this case would be exponentially small in the parameter ma.

For the spin-$\frac{1}{2}$ case for $\varkappa=0$ we find (Mamayev and Mostepanenko 1978a,b):
for $t\ll m^{-1}$

$$\epsilon^{(1/2)}=-\frac{q^2m^2}{8\pi^2t^2}\left(\ln\frac{1}{mt}+D^{(1/2)}\right)+\frac{F^{(1/2)}}{480\pi^2t^4} \tag{34}$$

for $t\gg m^{-1}$

$$\epsilon^{(1/2)}=K^{(1/2)}m^4(mt)^{-3q}+\frac{F^{(1/2)}}{480\pi^2t^4} \tag{35}$$

where $F^{(1/2)}=\frac{1}{2}q^2(11q^2-36q+18)$ and $D^{(1/2)}$, $K^{(1/2)}$ are other constants depending on q.

For the hyperbolic model ($\varkappa=-1$) there would be additional terms coming from the term $\varkappa/4$ in the integrals. These corrections are negligibly small for the realistic models of the Universe. The same is true for the analogous corrections in the spherical model.

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